APPENDIX A7.1 BASIC GEOMETRIC TRANSFORMATIONS (TWO-DIMENSIONAL)

A.7.1.1 Linear mapping or transformation of a point: $P \rightarrow P'$

A point is defined by its co-ordinates on each dimension: $P = (x, y)$. If two points are related by a linear equation, then one is said to be a *linear mapping* of the other:

$$\begin{cases} a_1x + b_1y = x' \\ a_2x + b_2y = y' \end{cases}$$

This defines the mapping of point $P = (x, y)$ into point $P' = (x', y')$. Put in another way, the coefficients a and b can be gathered into a *transformation matrix* **T** which, when applied to P, maps it into P':

$$\underset{\mathbf{T}}{\begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}} \underset{\mathbf{P}}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{\mathbf{P'}}{\begin{pmatrix} x' \\ y' \end{pmatrix}}$$

Example **Q′** is a linear mapping of **Q** (Figure A7.1)

$$\underset{\mathbf{T}}{\begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}} \underset{\mathbf{Q}}{\begin{pmatrix} 6 \\ 5 \end{pmatrix}} = \underset{\mathbf{Q'}}{\begin{pmatrix} 2.8 \\ 2.6 \end{pmatrix}}$$

A.7.1.2 Origin and translation

The *origin* of the space is normally defined as the point (0, 0), and all other points are defined by reference to this origin.

It is sometimes useful to move the origin of the space to a new position. This is termed *translation* of the origin (and axes). A particularly common translation is to the centroid—the average co-ordinate or centre of gravity—of a configuration of points. If the origin is moved to (a, b), then the co-ordinates of a point $\mathbf{P} = (x, y)$ relative to the *new* origin are given by: $\mathbf{P} = (x - a, y - b)$.

In the example below, if the origin is translated to (3, 4) then the point **P** has the co-ordinates (2, 1) in terms of the old origin and has the co-ordinates $(-1, -3)$ in terms of the new origin (Figure A7.2).

Note Translation preserves distances (identically).

Translation does *not* preserve scalar products or angular separation between vectors (see Appendix A2.1, Figure A2.3).

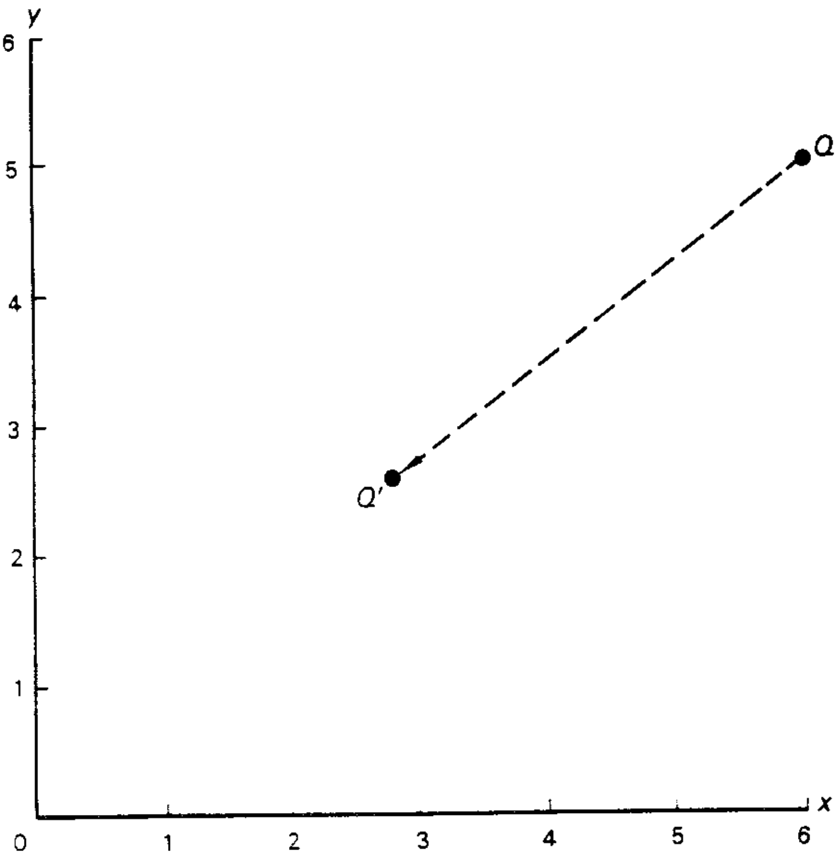


Figure A7.1 *Linear mapping*

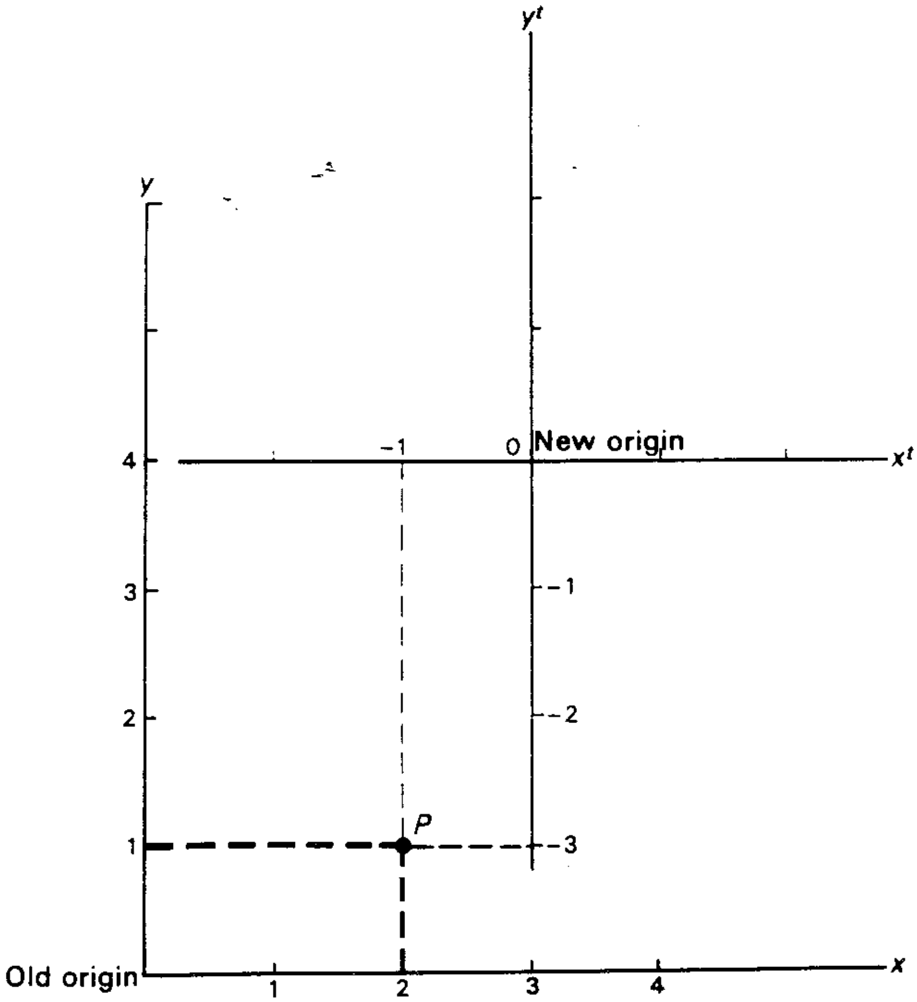


Figure A7.2 *Translation of origin and axes*

A7.1.3 Elementary transformations (null and unit)

Two very basic transformations (which serve as the geometric analogue of zero and unity in algebra) are the *null* and the *identity* transformations:

(i) *The null matrix:* $\mathbf{T} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

projects all points to the origin (0, 0) of the space and in so doing destroys all information (and obviously preserves neither distances nor scalar products),

e.g. $$\underset{\mathbf{T}}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}\underset{\mathbf{P}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{O} \text{ (origin)}$$

(ii) *The unit matrix:* $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

simply preserves the exact location of all points and is therefore an identity mapping. It preserves all information (including distances and scalar products),

e.g. $$\underset{\mathbf{T}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\underset{\mathbf{P}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{P}'$$

A7.1.4 Diagonal transformations (reflection and rescaling)

A particularly important set of geometric transformations involve diagonal transformation matrices (whose off-diagonal elements are zero). Clearly, the unit matrix is one example; other relevant instances are:

reflection;
uniform rescaling;
differential rescaling (weighting of axes);

(i) *Reflection*

Reflection of points in 2-space occurs when the sign (+, −) of *one* of the co-ordinate axes is changed. The effect is to 'flip' the configuration symmetrically. This is achieved by a diagonal transformation matrix, *one* of whose elements is −1.

Reflection in y-axis

$$\mathbf{T} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Reflection in x-axis

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note Reflection preserves distances and scalar products identically.

Example $P = (2, 1)$ and $Q = (4, 2)$ (see Figure A7.3)

(a) *Reflection in y-axis*

$$\underset{\mathbf{T}}{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}\underset{\mathbf{P}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \underset{\mathbf{P}'}{\begin{pmatrix} -2 \\ 1 \end{pmatrix}} \text{ and } \underset{\mathbf{T}}{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}\underset{\mathbf{Q}}{\begin{pmatrix} 4 \\ 2 \end{pmatrix}} = \underset{\mathbf{Q}'}{\begin{pmatrix} -4 \\ 2 \end{pmatrix}}$$

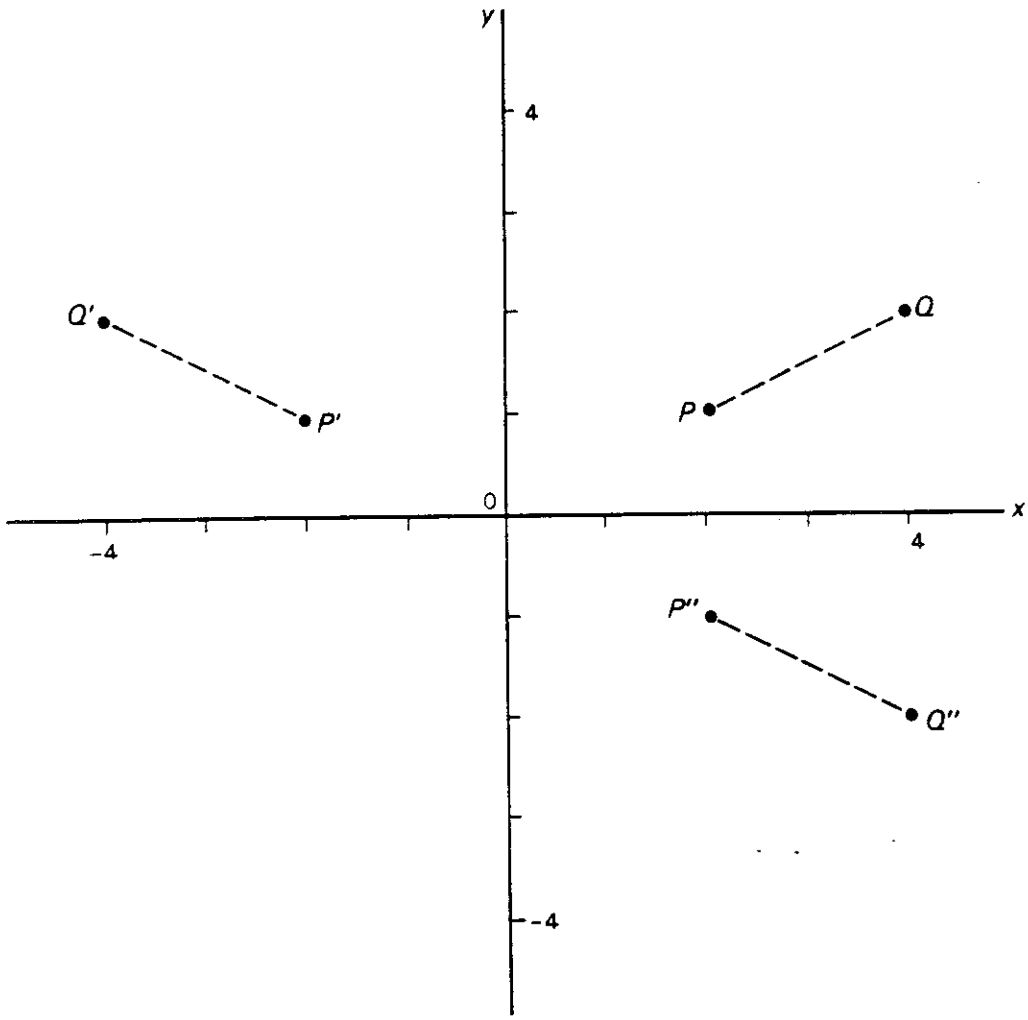


Figure A7.3 *Reflection on x-axis*

(b) *Reflection in x-axis*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\mathbf{T} \quad \mathbf{P} = \mathbf{P}'' \qquad \mathbf{T} \quad \mathbf{Q} = \mathbf{Q}''$$

(ii) *Uniform weighting (rescaling)*

A diagonal transformation matrix *with identical diagonal elements a,*

$$\mathbf{T} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

produces a uniform stretching ($a > 1$) or shrinking ($0 < a < 1$) of point locations, with each axis weighted by a factor of a. The effect is to rescale a configuration by a factor of a.

Note Uniform weighting/rescaling preserves *relative* distances and scalar products, up to a proportional rescaling by a, i.e. a ratio scale.

Example $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

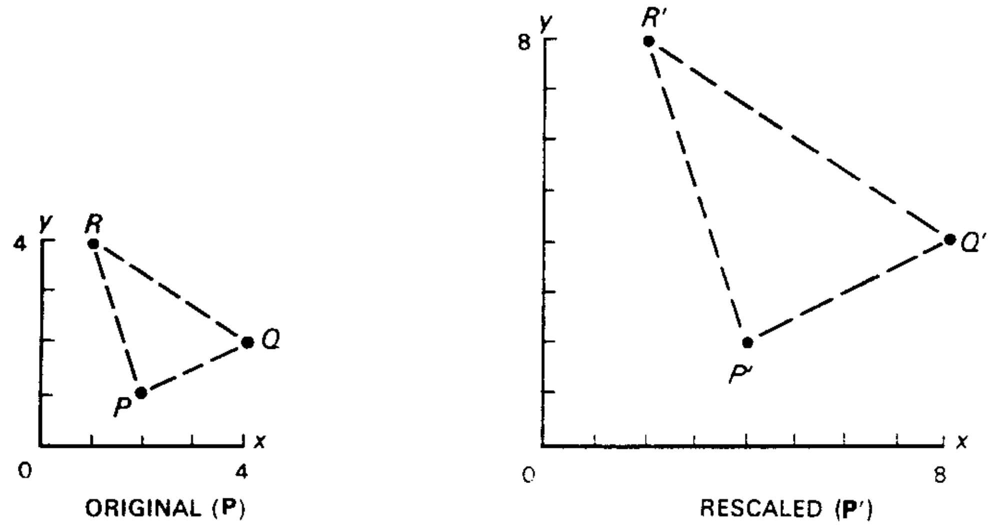


Figure A7.4 *Rescaling with uniform weighting*

and the point co-ordinates for P, Q and R are gathered into a matrix $\mathbf{P}$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \overset{\begin{matrix} P & Q & R \end{matrix}}{\begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \end{pmatrix}} = \overset{\begin{matrix} P' & Q' & R' \end{matrix}}{\begin{pmatrix} 4 & 8 & 2 \\ 2 & 4 & 8 \end{pmatrix}}$$

$$\mathbf{T} \quad . \quad \mathbf{P} \quad = \quad \mathbf{P}'$$

In the distance matrix below, the Euclidean distances in the *original* configuration ($\mathbf{P}$) are given beneath the diagonal, and those in the rescaled configuration ($\mathbf{P}'$) are given above the diagonal. Note that the distances are doubled, since $a = 2$.

$$\mathbf{D} = \begin{pmatrix} & P & Q & R \\ P & 0 & 4.47 & 6.32 \\ Q & 2.24 & 0 & 7.21 \\ R & 3.16 & 3.61 & 0 \end{pmatrix}$$

(iii) *Differential weighting (rescaling)*
A diagonal transformation matrix whose diagonal values are *not* equal

$$\mathbf{T} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{with} \quad a = b$$

produces a differential elongation or compression of the X and Y axes, with the X-axis weighted by a and the Y-axis by b. The effect is to distort the configuration along the directions of the axes, increasing the co-ordinate values if the weight is greater than 1, and decreasing them if it is less than 1. In the process of differential rescaling the original distances are changed, often dramatically so if the weights are very different.

Note Differential weighting does *not* preserve even relative distances or scalar products. It is not therefore a similarity transformation (see below).

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix} \overset{\begin{matrix} P & Q & R \end{matrix}}{\begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \end{pmatrix}} = \overset{\begin{matrix} P'' & Q'' & R'' \end{matrix}}{\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 3 & 6 & 12 \end{pmatrix}}$$

$$\mathbf{T} \quad \mathbf{P} \quad = \quad \mathbf{P}''$$

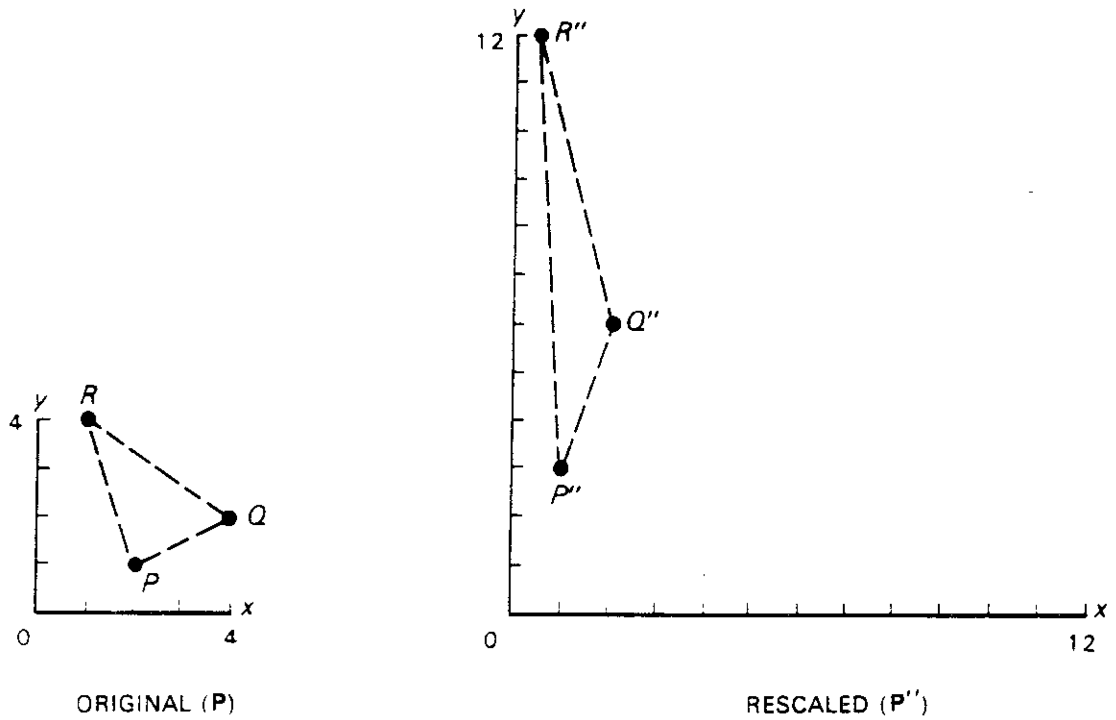


Figure A7.5 *Rescaling with differential weighting*

$$\mathbf{D} = \begin{array}{c} \\ P \\ Q \\ R \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{pmatrix} 0 & 3.16 & 9.01 \\ 2.24 & 0 & 6.19 \\ 3.16 & 3.61 & 0 \end{pmatrix} \end{array}$$

(original distances below diagonal, rescaled above diagonal)

A7.1.5 (Orthogonal) rotations

Very often the co-ordinate axes of a set of points are arbitrary, especially for Euclidean distance calculations, and another set of co-ordinate axes (such as principal components) may be preferable or have more desirable properties. The move from one set of co-ordinate axes to another is termed a *rotation of axes* about the origin of the space. In the 2-dimensional case, this move is described in terms of the *direction* and the *extent* of the change. (Here we assume a rigid or *orthogonal* rotation, keeping the axes at right angles.) The axes may be moved clockwise, or in an anticlockwise direction, and the extent of the rotation is given by the angle θ through which the axes move.

(i) *Permutation of axes.*
A rigid rotation anticlockwise through 90° is effected by a rotation matrix of the form $\mathbf{T}_a$, (where the subscript a denotes an anticlockwise rotation)

$$\mathbf{T}_a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This has the effect of permutating the x into the y axis, or moving the co-ordinate system through a right angle. The rotation matrix, $\mathbf{T}_c$, (where the subscript c denotes a clockwise rotation)

$$\mathbf{T}_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

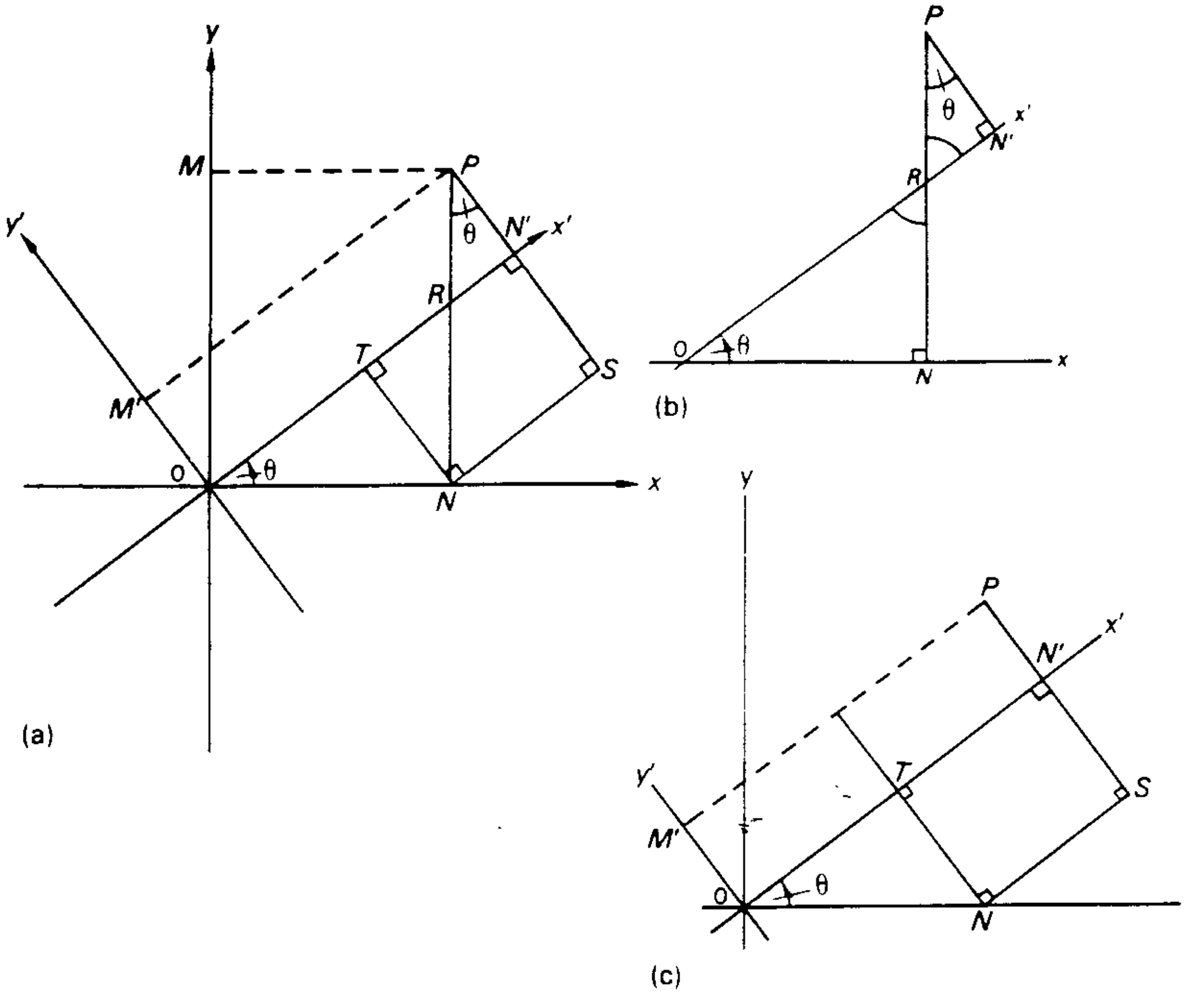


Figure A7.6 *The geometry of rotation of axes*

has the opposite effect; rotating the axes in a clockwise direction through 90°.

(ii) *Rotation through an angle*
For any angle other than 90°, the anticlockwise rotation matrix $\mathbf{R}_a$, taking the original x, y axes into the new x', y' axes has the form:

$$\mathbf{R}_a = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where θ is the positive angle through which the original reference axes are rotated. (Readers who wish to ignore the trigonometry of this rotation matrix should skip the following section and move to the example, simply noting that if $\theta = 90°$, then $\cos\theta = 0$ and $\sin\theta = 1$, reducing $\mathbf{R}_a$ to the simple form $\mathbf{T}_a$ noted earlier.)

How is the rotation matrix derived? Although not obvious, the procedure is quite simple. You are advised to follow the steps in Figure A7.6 (a)–(c).

First we do some elementary geometry.

(1) We identify a point P with reference to the old axes (x, y) and the new axes (x' y') which have been rotated through an angle of θ degrees.

(2) The co-ordinates of P on x and y are N and M respectively, its co-ordinates on x' and y' are respectively N' and M'.

(3) It is useful to convince yourself at this stage that the length OM = the length PN and, similarly $MP = ON$, $OM' = N'P$ and $ON' = M'P$.

(4) We label the point where PN crosses x' as R for convenience, and consider the two triangles ORN and RPN' (see inset b). We know that the angles of a triangle sum to 180°, and therefore the angle ORN must be $180° - (90° + \theta°) = 90° - \theta°$. The angle PRN' is also $90° - \theta°$, thus the angle RPN' must be $\theta°$. We will make use of this fact later.

(5) We now construct a rectangle $TN'SN$, redrawn for clarity in inset c. Again the reader should be convinced that the length ON' is equal to $OT + TN'$ and ON' *is also equal to* $OT + NS$ (since $TN' = NS$ by construction).

(6) Also, OM' is equal to $PS - SN'$ and OM' *is also equal to* $PS - TN$ (since again $SN' = TN$ by construction).

(7) Returning to the large diagram (inset a) we are now in a position to give expressions for the new co-ordinates of P (N' and M') in terms of the old co-ordinates (N and M), thus

$$ON' = OT + NS$$

and $$OM' = PS - TN$$

(8) Now consider the triangles OTN and PNS. The reader should be convinced that these right-angled triangles are similar, having identical angles (90°, θ and $90° - \theta°$).

(9) We know that in a right-angled triangle the cosine of an angle θ is given by the ratio of the side adjacent to the angle to the hypotenuse: Thus, in triangle OTN the cosine of θ ($\cos\theta$) is OT/ON, and in PNS, $\cos\theta = PS/PN$.

(10) Similarly, we know that the sine of an angle ($\sin\theta$) is given by the ratio of the side opposite the angle to the hypotenuse. Thus, in PNS the sine of θ ($\sin\theta$) is NS/PN, while in OTN, $\sin\theta = TN/ON$.

(11) We are now in a position to do some algebra. Consider the fact that

$$ON' = OT + NS \tag{1}$$

We seek an expression for ON' in terms of O, N, M, P and the angle θ (i.e. an expression which will relate the new axes to the old).

We know that

$$\cos\theta = \frac{OT}{ON}.$$

Cross-multiplying gives

$$OT = ON\cos\theta. \tag{2}$$

Similarly we know

$$\sin\theta = \frac{NS}{PN}.$$

Again, cross-multiplying gives

$$NS = PN\sin\theta. \tag{3}$$

Thus, substituting (2) and (3) back in (1), we obtain

$$ON' = ON\cos\theta + PN\sin\theta. \tag{4}$$

We may treat the expression

$$OM' = PS - TN$$

in precisely the same way.

$$\begin{aligned}
\cos\theta &= \frac{PS}{PN} \\
PS &= PN\cos\theta \\
\sin\theta &= \frac{TN}{ON} \\
TN &= ON\sin\theta
\end{aligned}$$

Therefore
$$OM' = PN\cos\theta - ON\sin\theta \qquad (5)$$

Thus in (4) and (5) we have derived the desired result: an expression for the new co-ordinates in terms of the old co-ordinates and the angle of rotation.

Now, letting x stand for ON, y for OM, x' for ON' and y' for OM' we may write

$$\left.\begin{aligned}
x' &= x\cos\theta + y\sin\theta \\
y' &= -x\sin\theta + y\cos\theta
\end{aligned}\right\}$$

which in matrix form is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.

$$\mathbf{x}' = \mathbf{R}_a\mathbf{x}$$

which is the desired result.

An example will illustrate the procedure. In Figure A7.6a. the angle of rotation ,θ, is almost 37°*. In this instance, the position of P with respect to the new axes (X', Y') is obtained as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 37^\circ & \sin 37^\circ \\ -\sin 37^\circ & \cos 37^\circ \end{pmatrix} \begin{pmatrix} 4.0 \\ 5.2 \end{pmatrix} = \begin{pmatrix} 6.32 \\ 1.75 \end{pmatrix}$$

If we wish to rotate a whole configuration of P points in 2 dimensions, the same rotation matrix is applied to each point. (In more than 2 dimension the rotation procedure is slightly more complicated, but consists of taking all pairs of dimensions and proceeding in the above manner.)

A7.1.6 Transformations: a summary

In discussing the various geometrical transformations employed in MDS, the following are particularly significant and basic:

The *identity or unit* transformation, which leaves a configuration and its absolute and relative distances and scalar products unchanged;

The *(central) dilation or rescaling* transformation, which multiplies each axis by a

*In fact it is 36°52′; the triangle ORN is a 3:4:5 triangle, yielding a cosine of 4/5 and a sine of 3/5.

given scalar value. It preserves relative (but not absolute) distances and scalar products.

The *differential dilation or rescaling* transformation, which multiplies each axis by a different scalar value. It preserves neither relative nor absolute distances (nor scalar products).

The *translation* transformation which removes the origin of the space. It preserves relative and absolute (Euclidean) distances, but changes scalar products.

The *orthogonal rotation* transformation, which preserves Euclidean distances and scalar products identically.

An *extended similarity* transformation (often called a similarity transformation) is a composite transformation, which preserves all relative distances. It may comprise a reflection, a translation of origin, a central dilation and an orthogonal rotation. Taken together these are permissible operations on a distance model configuration.

7.3 Comparing Configurations

When using MDS, it is only a matter of time before the user wishes to compare two or more configurations. If a study has set out to replicate a previous one, then it is important to know the ways, and the extent, to which the current MDS solution resembles that of the original study. More often, the user has employed more than one variant of MDS on the same data, or has scaled the data of different subgroups of subjects, and the issue of similarity between the resulting configurations once again arises.

Before attempting to compare configurations the user should be sufficiently persuaded that in each case the fit of the solution to the data is good enough to warrant proceeding any further.

The answer to the question, 'How similar are two configurations?' depends on two things:

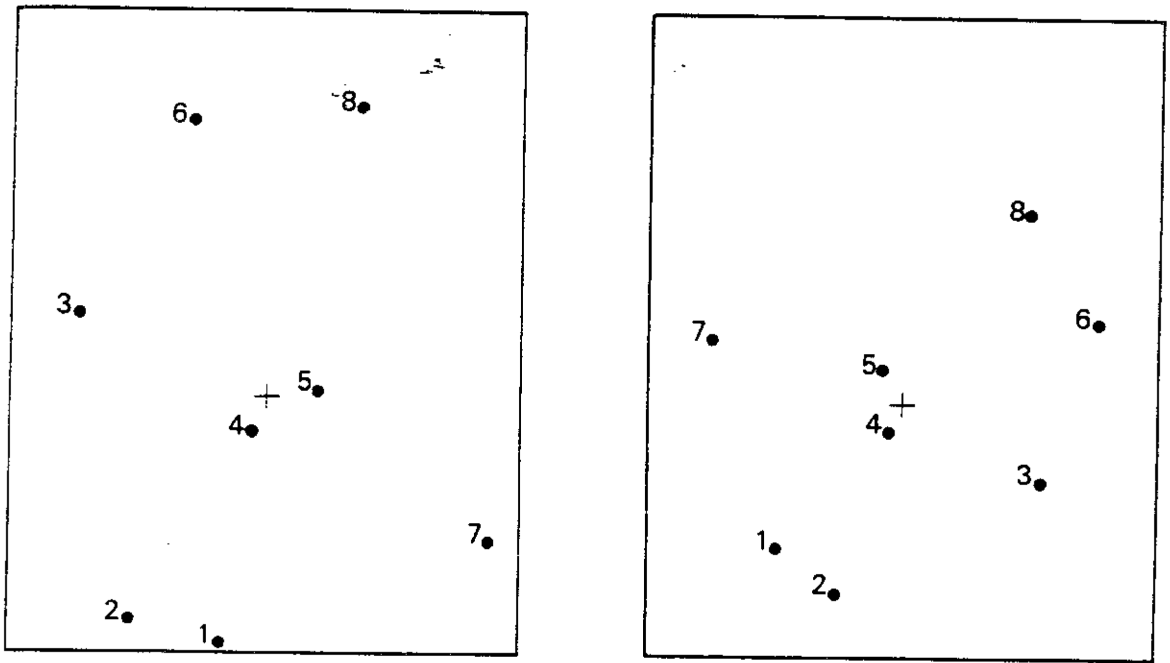
(i) what aspects of the configuration are considered *relevant*, and
(ii) what properties of the configuration are *unique*.

*Latent structure is a family of models developed to analyse sets of dichotomous response patterns (Lazarsfeld and Henry 1968; Fielding 1977). In these models, observed patterns are thought of as arising from the multiplicative combination of the latent probabilities of giving a positive answer to each item, where these probabilities differ according to the position of the subject in the space. In the latent class model, the 'space' consists of a set of partitions or classes. Whilst the ideas of latent class analysis are very appealing, estimation of the model parameters have proved to be very difficult. However, Carroll (1975) shows how the latent class model may be seen as a special case of canonical decomposition, where the dimensionality may be identified with the number of latent classes and the parameters of the model may be estimated in a straightforward manner from the CANDECOMP weights. Paradoxically, the CANDECOMP estimation is better than the procedure suggested by Lazarsfeld and Henry, despite the fact that it is minimising a different and less obviously appropriate badness-of-fit function.

For instance, if one configuration were simply twice the size of the other but in every other way identical, it is unlikely that most researchers would consider this a relevant difference. Hence one would normally wish to compare *relative* rather than absolute distances. If dealing with solutions obtained from a Euclidean distance model, it is unlikely that the orientation of the configuration will be considered relevant since any rigid rotation leaves distances unchanged. Similarly, the origin of the space would normally be treated as arbitrary, unless the model were a vector model (since change of origin alters scalar products and hence alters the angles separating the vectors, cf. Appendix A2.1) or unless a facet analysis had been employed in interpretation and the point chosen as the origin was substantively meaningful.

In actual fact, many users of MDS and factor analysis resort to crude and misleading methods for comparing configurations—such as restricting attention to the first two dimensions of a solution and simply 'eye-balling' the configurations, hoping that salient differences will in some way reveal themselves. Unfortunately, differences which are irrelevant can often make identical configurations *appear* to be very different. This is illustrated in Figure 7.4, where the two-dimensional configuration of seriousness of offences given in Figure 1.1a, is first reproduced as Figure 7.4a, and then submitted to a series of transformations which preserve relative distances. In terms of keeping the same relative distances these two configurations are identical but they certainly *look* different. So appearances are not a reliable guide as to how two configurations are alike. How, then, does one go about comparing them?

Sometimes transparent acetate sheets are used to compare two-dimensional configurations—one configuration is first copied onto a transparent sheet and laid



(a) (Fig. 1.1)

(b): (a) subjected to restricted similarity transform:
(i) clockwise rotation through 40°
(ii) reflection in axis I
(iii) rescaling by factor of 0.75

Figure 7.4 *Two 'identical' configurations* on top of the other configuration. By moving the sheet in a circular manner (rotations) and/or flipping it over (reflections) the one configuration can be moved into maximum apparent congruence with the other. This procedure has its uses, but even this expedient cannot deal with differences of scale, and is obviously restricted to two-dimensional situations. Moreover, we still need some explicitly defined index of configurational similarity if the comparison is to be anything more than approximate or if we need to compare more than two configurations. How, then, should two or more configurations be compared?

(i) First, we must return to the question of what aspects count as significant and unchangeable information and what are irrelevant.
(ii) That decided, a method is needed which will bring the configurations into the closest possible conformity with each other.
(iii) Finally, some indication is needed of how closely the configurations correspond to each other, preferably by using an appropriate measure of goodness of fit.

We shall take up each of these points in turn.

First of all, in this section (and indeed, as far as 7.4.1) we will assume that two configurations are considered identical if they only involve differences of:

(1) *scale* (how large the actual configuration is);
(2) *orientation* (rigid rotation and/or reflection of axes); and
(3) *origin* (the zero point of the space).

Consequently, configurations may be shrunk or expanded at will (1), moved—rigidly rotated—through any angle (2), and may have the origin translated to any point in the space (3) in order to get them into greater conformity with each other. The value of any index of similarity between configurations should remain unchanged whenever these operations are performed.

7.3.1 Geometric transformations of a configuration

The geometry of these three basic operations, which taken together define an 'extended similarity transformation', is described in Appendix A7.1. Users who are unfamiliar with them should read it carefully before proceeding further. Two transformations are particularly important in examining the similarity between configurations and in moving them into closest conformity. Both keep the comparative or relative distances in the configurations unchanged:

Configuration transformations which preserve (Euclidean) distances
1 *Extended similarity:* involves rotation, translation and rescaling
2 *Restricted similarity:* involves rotation and rescaling (no translation of origin)

Usually it will not be possible to transform two (or more) configurations into an identical structure, and it is therefore necessary to define an index of how similar two configurations are. All commonly encountered indices can be thought of as being a function of the *distances* between the points in the two configurations. Gower (1979, 1980) has provided an excellent review of such indices by examining the form of the function relating the distances in the two configurations. One obvious measure for assessing the similarity between two configurations is the product moment correlation* between the distances involved, and this is commonly used. A related measure of similarity between the configurations is called S by Lingoes (Lingoes and Schönemann 1974, p. 436). This has the nice properties that its value depends neither upon the number of points, nor upon the number of dimensions, nor on the scale of the configuration. Consequently it can be used to answer the question, 'Does configuration X match Y better than X matches Z?', which will be a central issue when we come to compare several configurations. An even simpler measure, which is just the squared linear correlation r^2 between the co-ordinates of the configuration X and Y which have been brought into maximum conformity, is also frequently used. It is directly related to S and shares its desirable properties.†

7.3.2 Comparison using Procrustes analysis

Procoptes, better known as Procrustes, 'The Smasher', caught (travellers) on the borders of Athens and by stretching or pruning made them an exact fit to his lethal bed.

(Kirk 1974, p. 153)

As originally employed in comparing configurations, simple Procrustes analysis consisted of moving two configuration matrices **X** and **Y** into closest conformity, allowing only rotation and reflection. Procrustes analysis was later extended to include rescaling and translation of origin. This more extended usage, which will be employed here, is termed generalised Procrustes analysis (Gower 1975) and is illustrated in Figure 7.5. Here three configurations (A, B, C) of four points each forming a roughly rectangular shape have been positioned into closest conformity with each other and, since the configurations are not identical, a new configuration **Z** is then produced, defined by the location of the square points in Figure 7.5. This new configuration is an 'average configuration' in the sense that each of its points is a least-squares fit to the corresponding points of the original configurations. This best fitting configuration is called by a variety of names: the compromise, consensus, centroid (as in PINDIS) or group configuration (as in INDSCAL). From now on it is referred to as the centroid configuration, and denoted by **Z**.

The iterative computing procedure for producing the centroid configuration is described in Gower (1975, p. 43).

Procrustes rotation can be used on any number of configurations, and is commonly thus used.

7.4 Procrustes Rotation and Individual Differences Scaling

Procrustes rotation is not only useful for comparing configurations, it can also serve as a basis for individual differences scaling. The basic idea is simple. Given a set of configurations, we begin by moving them into closest conformity by

*Carroll (1972) has suggested the use of linear, monotone, non-linear (continuity) and canonical correlation coefficients for assessing different aspects of goodness of fit between two configurations, according to different criteria for assessing 'fit'.

†$S^2 = 1 - r^2(X, Y)$ is the equation relating the two measures, and both measures remain unchanged when a similarity transformation is performed on X and Y to bring them into maximum conformity.

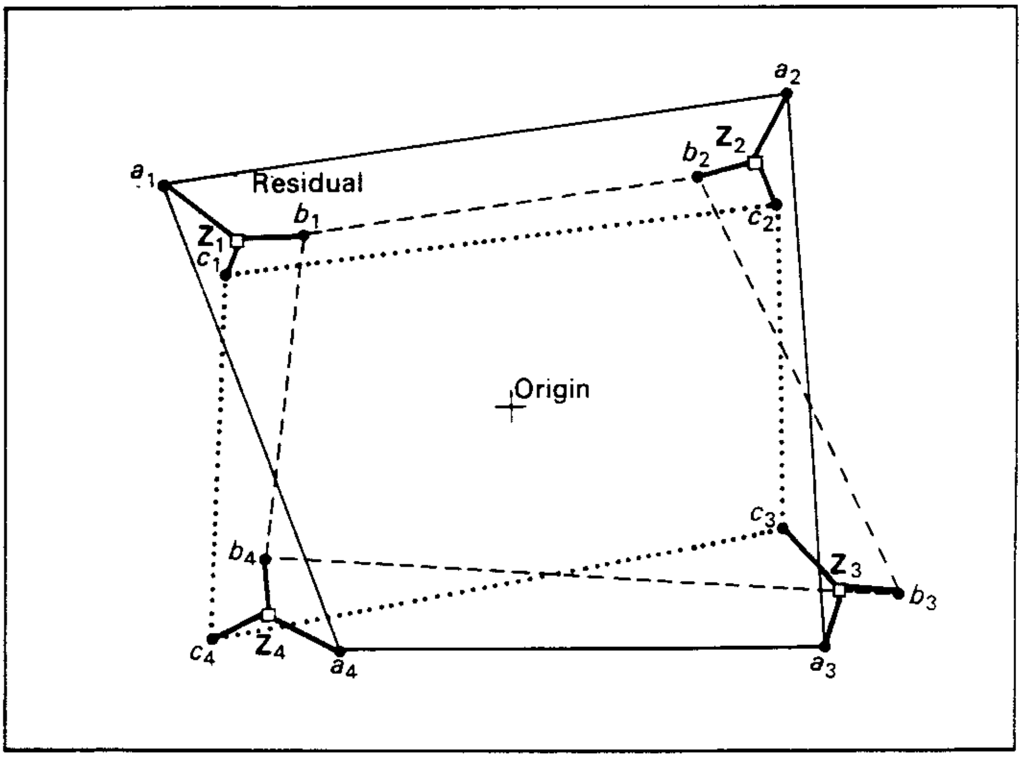
generalised Procrustes analysis, thereby producing the centroid configuration. Next we measure how closely each configuration fits the centroid configuration. Conceptually, the *badness* of fit of each configuration to the centroid can be defined in terms of the sum of squares of the residuals, i.e. $\Sigma (x_i - z_i)^2$ in Figure 7.5, but we shall follow Lingoes' example and use the squared correlation between the 'subject' configuration co-ordinates and those of the centroid configurations as a measure of *goodness* of fit.

7.4.1 *The* PINDIS *models*

Lingoes has developed a series of increasingly complex models based on Procrustes rotation, which have affinities with other models we have encountered. His set of models is known collectively as PINDIS (Procrustean INdividual DIfferences Scaling).

The data for PINDIS, unlike other three-way models, consist of a *set of configurations* obtained from previous scaling solutions. The configurations are first moved into maximum conformity by generalised Procrustes rotation. Since this procedure consists entirely of 'admissible' transformations (in the sense that they leave the relative distances unchanged) this basic general similarity 'model' (denoted P0) provides a yardstick or reference point for subsequent models which do *not* leave original relative distances intact.

Beyond P0, the models all involve so-called 'inadmissible' transformations—



(Adapted, with permission, from Gower 1975)

3 configurations of 4 points, with a common origin: A denoted by ——— B denoted by – – – – C denoted by ··········

Best fitting Procrustes ('centroid') configuration is given by **Z** (z_1, z_2, z_3, z_4)
It minimises sum of squares of residuals (denoted by thick lines).

Figure 7.5 *Procrustes analysis of 3 configurations* i.e. operations which change the original relative distances in some systematic way in order to obtain a better fit between the original configuration $\mathbf{X}_i$ and the reference centroid configuration $\mathbf{Z}$. At this point it should be stressed that:

(i) the centroid configuration is defined somewhat differently in each model unless the user wishes to keep it unchanged throughout (in this latter case, PINDIS becomes an 'external' analysis, an option effected in the MDS(X) version by using the READ HYPOTHESIS command);

(ii) at each stage (in each model) the individual configurations are moved into optimal fit to the centroid configuration, using admissible transformations.

The interrelations between the PINDIS models are illustrated in Figure 7.6. A brief simplified resumé will be given here, and separate models are discussed in greater detail in subsequent sections. In the diagram, each model is represented by a box. The top half indicates the usual name of the model, and the bottom half indicates what operations are performed on the centroid matrix (Z) and what 'individual differences' parameters are estimated in the model. Arrows are drawn upward from less general to the more general models. Note that the models do *not* form a strict order, but rather two parallel hierarchies, i.e. the *distance* models (P2, P1, P0) and

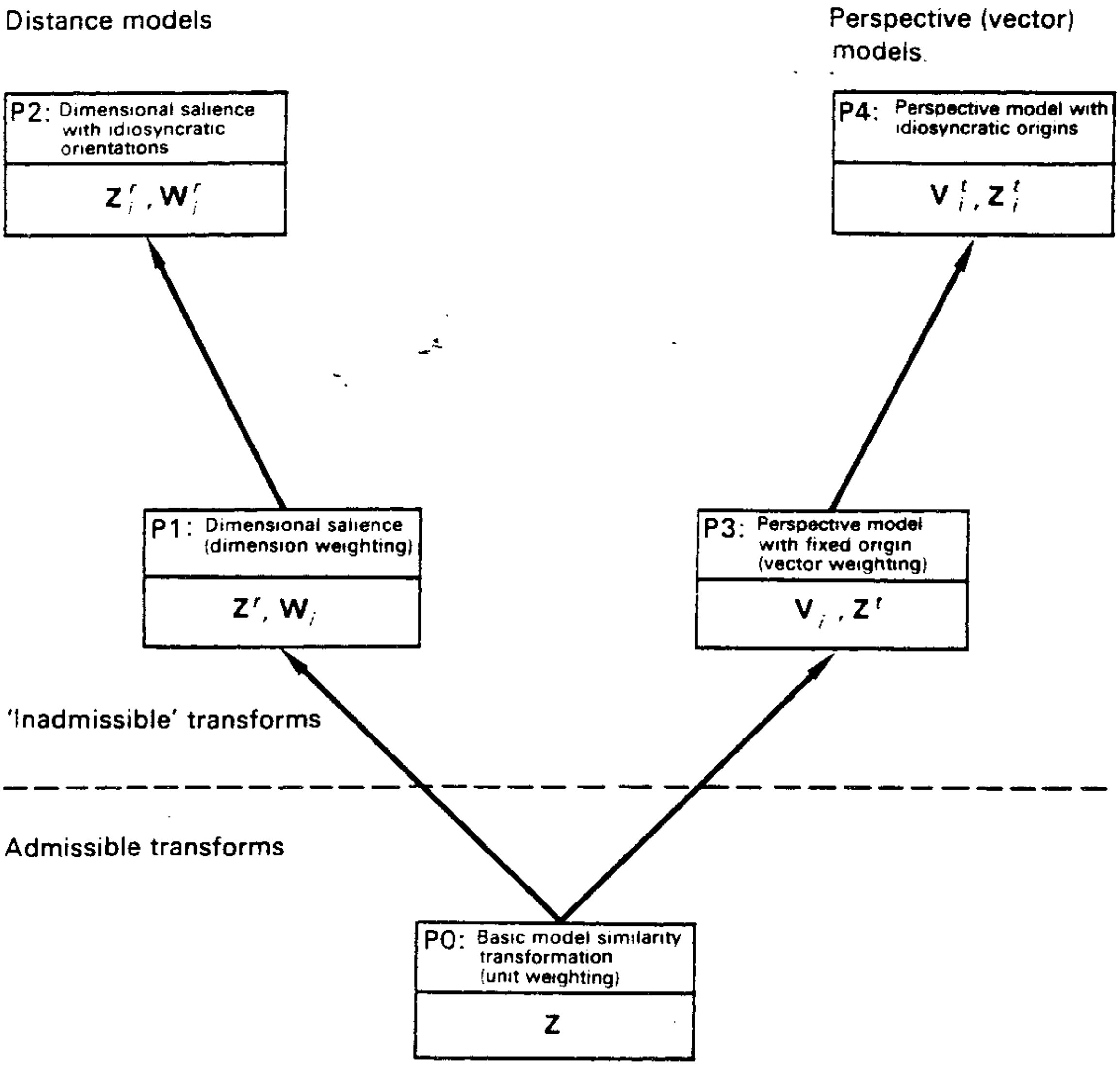


Figure 7.6 *PINDIS models*

the *perspective/vector* models (P4, P3, P0), which share a common basis (P0).*

Before going on to describe the models, some preliminary points need to be made about the form of their specification (in the lower half of the boxes in Figure 7.6).

(i) The parameters of each model are of two sorts—they refer either to what is done to the centroid matrix, **Z**, and to what systematic weights (dimensional weights, **W** or vector weights, **V**) are applied to move the centroid configuration into greater conformity with the original configurations.

(ii) The irritating superscripts and subscripts which bedeck the matrices in the model specification are in fact necessary to distinguish between the models, and repay careful attention. The original average centroid configuration Z appears in its pristine, undecorated form in P0. In every other model the centroid configuration is changed in some way.

In the distance models, the superscript r signifies that it is rotated. Thus the centroid configuration is rotated in both P1 and P2—to a different orientation for each individual configuration in P2 (signified by $\mathbf{Z}_i^r$) and to a *single* new orientation in P1 (signified by $\mathbf{Z}^r$), indicated by the absence of a subscript. In the vector models, the superscript t denotes translation or shift of origin. Hence the centroid configuration is translated to a unique position for each individual in P4 ($\mathbf{Z}_i^t$) and to a single new origin in P3 ($\mathbf{Z}^t$).

The Distance Models

P1 The dimensional salience model is the PINDIS equivalent of INDSCAL. The centroid configuration is first rotated into an optimal position for dimensional weighting and then a set of individual dimension weights are estimated for each input configuration. These weights are entirely analogous to INDSCAL weights, except that in PINDIS they may legitimately be negative, in which case they signify the reflection of the dimension concerned.

P2 The individually rotated dimensional salience model is the PINDIS equivalent of the Carroll-Chang (Carroll and Wish, 1973, pp. 90 et seq.) IDIOSCAL model which allows the axes of the centroid configuration to be rotated to an idiosyncratic orientation for each individual configuration, and then be differentially weighted. In this model, the dimension weights will only be comparable between 'subjects'/configurations if they happen to share the same rotation.

The Vector Models

P3 In the basic simple perspective model, the origin of the centroid configuration is first translated to an optimal position and a vector is then constructed from the origin to each of the constituent stimulus points. Each individual configuration can be thought of as having had a different set of weights applied to each of the vectors. A high vector weight will push a point further out from the origin, and a low vector weight will contract the vector and move a point closer to the origin. A negative

*A further 'double-weighted' model is discussed by Lingoes and is available in the PINDIS program. However, this model is particularly subject to sub-optimal solutions and its use is not generally recommended. Its estimation is suppressed in the MDS(X) version by setting SUPPRESS(1). The double-weighted model is not treated further here.

weight (which is quite permissible) has the effect of 'flipping' the vector in the opposite direction. The effect of a set of vector weights is thus to 'unscramble' a configuration by selectively relocating the stimulus points of the centroid configuration, with the proviso that they can only move in the same direction, towards or away from the origin.

P4 The individually translated perspective model differs from P3 in allowing *each* individual configuration to have its own 'point of view' (origin). Since, as we know, translation of origin changes vector separations, the same set of vector weights may well have markedly different effects on differently centred configurations in this model. Thus P4 permits each individual configuration to have a different origin *and* a different set of weights. It is just as well that Procrustes' imagination was not fired by modern MDS!

Psychologically, the vector models have a certain appeal, since they allow different *categorisations* of stimuli to be related by regular transformations, and allow for such processes as over-compensation—in these models, the Maoist and the Stalinist can in effect share the same political map whilst consigning each other to the fascist camp by means of a single parameter, the negative vector weight!

These increasingly complex transformations bring better fit, but at a cost. In its more complex models PINDIS becomes prolific in its use of parameters, and many users (especially statisticians) are rightly wary of the degrees of freedom consumed. The parsimony of Occam's razor appears not simply to be blunted but to be thrown away with abandon. Assuming that the number of stimuli considerably exceeds the number of dimensions, then the models assume a natural hierarchy defined by the number of parameters (cf. Lingoes and Borg 1978, p. 495).

Fewest free parameters	*Model*	*Parameters per individual configuration*
	P0	(No parameters) (only admissible transformations)
	P1	r dimensional weights
	P2	r dimensional weights and $\binom{r}{2}$ rotation coefficients
	P3	p vector weights
	P4	p vector weights and translation vector of r elements
Most parameters	(P5	r dimension weights and p vector weights)

When deciding which PINDIS model is most appropriate, one will necessarily be trading off the increase in goodness of fit (or explained variance) against the increase in the number of fitting parameters. There are no reliable statistical ways for deciding whether the trade-off is worth it, so the assessment of which PINDIS model is best will always retain a strong subjective element.*

7.4.2 The distance models (P1 and P2)

The two forms of distance (or, more strictly, dimensional) model in PINDIS examine the extent to which a given configuration can be better fitted to the centroid

*Since this was written, Langeheine (1980) has produced the results of his simulation studies of the PINDIS models, which provide expected fit measures and other statistics as a guide to help the user in deciding on the appropriate model. The use of these approximate norms is strongly recommended.

configuration by differential weighting of the axes (P1) or by differential rotation followed by individual weighting in the case of P2. It should be remembered that in neither case are the original relative distances preserved and in this sense the models involve 'inadmissible transformations'.

7.4.2.1 P1, the weighted distance model (with fixed dimensional orientation)
The specification of this model is formally identical to INDSCAL:

$$\delta_{jk}^{(i)} = d(x_j, x_k) = \sqrt{\sum w_a^{(i)}(z_{ja} - z_{ka})^2}$$

i.e. the Euclidean distance between points x_j and x_k in the original configuration $\mathbf{X}_i$ is assumed to be (a similarity transformation of) the distance between points z_j and z_k in the centroid configuration, after each dimension has been differentially weighted. A detailed comparison of P1 and the INDSCAL model is contained in Borg and Lingoes (1978).

The chief advantage which P1 has over INDSCAL is that in the PINDIS hierarchy it is possible to investigate first how much variation can be accounted for in a given set of data by legitimate similarity transformations (i.e. the Procrustes rotation of model P0) *before* having recourse to individual weights. Only if the improvement in explained variance between P0 and P1 is substantial is it worth proceeding to a more complex distance model. P1 differs from INDSCAL in two other significant ways:

(i) in the manner of estimating the group space, and
(ii) in the interpretation of the subject space weights.

In INDSCAL, the group space co-ordinates and subject weights are determined simultaneously, whereas in P1 the centroid space is determined first, then put into optimal orientation for dimensional weighting. Only then are the subject weights calculated as a separate operation. This difference has the effect of improving the properties of the subject space. In particular:

> The squared length of the vector drawn from the origin to the subject space point corresponds *exactly* in the P1 model of PINDIS to the variation in the subject's data (individual configuration) explained by the model and is independent of any orthogonality properties of the 'group space' or centroid configuration. (This, it will be recalled, holds only 'approximately' in INDSCAL, depending on the correlation between the dimensions.) It also means that in P1 it is possible to estimate the contribution of each dimension to the total communality, if the user so desires.
>
> The separation of subject vectors in the subject space correctly represents the correlation of the respective configurations in P1. (This holds only approximately in INDSCAL.)
>
> P1 can accommodate negative weights, which can be interpreted as reversed or reflected dimensions.

In actual practice, the application of P1 and INDSCAL to the same data (after preliminary scaling in the latter case) will lead to very similar results in terms of the centroid configuration, but there will often be significant differences in the subject space. This is well illustrated in the Borg and Lingoes 1978 paper.

An example involving the P1 model is given in 7.4.5.

7.4.2.2 P2, the idiosyncratically rotated and weighted distance model

This model allows individual configurations ('subjects') to differ both in the frame of reference which they adopt (i.e. in the dimensions they choose as significant, so long as they are orthogonal to each other) and in the weights they attach to them. This is also a dimensional model, but one where an idiosyncratic rotation of the axes of the centroid space occurs before the application of weights. The individual differences in rotation could be trivial or substantial. In most cases using PINDIS, it will be found that a slightly different individual orientation of axes will fit an individual configuration somewhat better than the averaged orientation provided by P1. (Indeed, in the PINDIS computational procedure, the individual orientations of axes are estimated first and the orientation for P1 is then averaged from these.)

On the other hand, some subjects may employ a rotation which is clearly different from others. In psychological terms, this often means that some *combination* of the initial P1 dimensions or properties are more salient than the original ones. If, for instance, it turns out that in the judgment of the Irish political candidates two main consensual dimensions are a pro/anti-British and a left/right dimension, it may well be that for a significant fraction of the population a single left-wing, anti-British *vs* right-wing, pro-British dimension is virtually the only one that matters, with small residual variation on the other dimension. In terms of the P2 model, this would mean that for such people a rotation of about 45° with a large weight on the new dimension I and a small weight on dimension II would provide a better frame of reference than the P1 centroid configuration.

The P2 model is very similar to the Carroll-Chang IDIOSCAL model (the acronym stands for Individual Differences In Orientation SCALing) to a variant of Tucker's Three-mode Scaling and to Harshman's (1972) PARAFAC-2. These are discussed in Carroll and Wish (1973, p. 440 et seq.) and the precise relationship between IDIOSCAL and INDSCAL is discussed in Borg (1979, p. 635 et seq.). The P2 and IDIOSCAL models resemble each other in much the same ways as P1 and INDSCAL do, in particular in having separate *vs* simultaneous estimation of the group space and subject parameters, clearer interpretation of the subject parameters in P2, and estimation from the original data (IDIOSCAL) as opposed to ready-scaled data (P2). These characteristics are discussed in Borg and Lingoes (1978) and Borg (1980), and both include a detailed mathematical treatment and a number of illustrative examples.

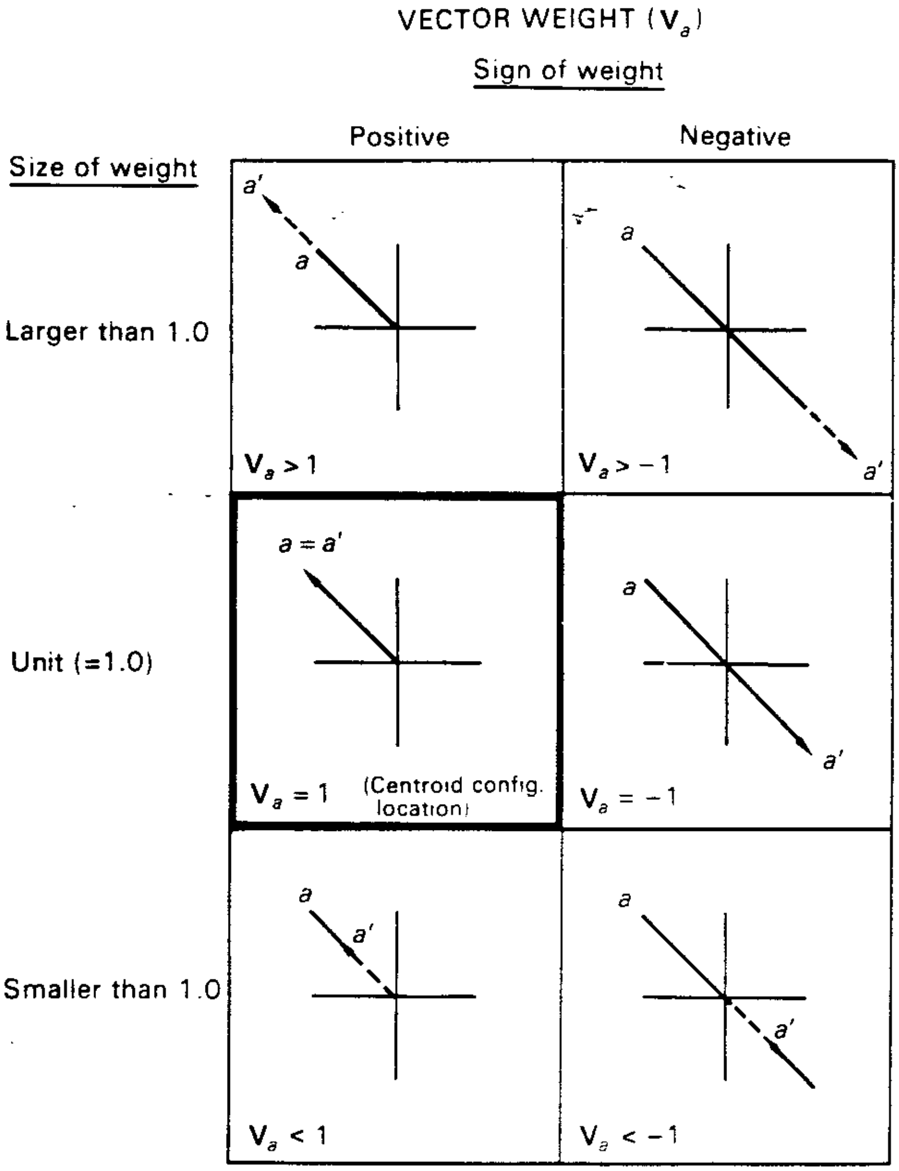
In many ways the P2 IDIOSCAL model is very appealing, but it is a rather complex and relatively ill-understood model, and one for which there are not as yet any very compelling empirical examples. Users are once again cautioned to proceed with care.

7.4.3 The perspective (vector weighting) models (P3 and P4)

The procedure involved in the perspective models is the construction of a vector from the origin to each stimulus point of the centroid configuration, and the weighting of these vectors differentially for each configuration in order to get it into

VECTOR WEIGHT: Size	*Direction* POSITIVE sign (same direction)	NEGATIVE sign (opposite direction)
LARGER (further out)	Point moves further out in same direction	Point moves a greater distance but away from origin
UNIT (same length)	*Point in same position as in Centroid Configuration*	Point is same distance from origin, but in opposite direction compared to Centroid
SMALLER (closer in)	Point moves closer towards origin	Point moves a smaller distance but away from origin

Table 7.3 *Effect of size and direction on point relocation*



Note: *a* denotes location of a point in the centroid configuration;
a' denotes relocation after applying vector weight (V_a).

Figure 7.7 *Effects of size and sign of vector weight (V_a) on point relocation* better fit with the centroid. Taking the original centroid vectors as the unit, each individual vector weight may be smaller, the same or larger, and it may be positive or negative in sign. These differences and their effects are summarised in Table 7.3 and illustrated in Figure 7.7. Note that whatever value the vector weight has, it can only relocate a point in the same or in the opposite direction.

7.4.3.1 P3, the weighted vector model (fixed origin)

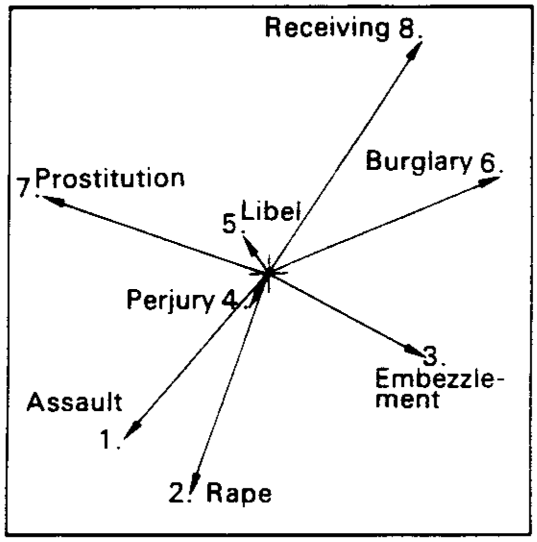
In P3 the main focus of interest is obviously on the set of vector weights which consistently transforms the centroid configuration into as close an approximation to this individual configuration as possible. What should be looked for in comparing individual sets of vector weights? For any configuration, the closer these weights are to +1, the more that individual configuration resembles the centroid and the less useful the vector model is. Within an individual set of weights interest will normally centre upon which are largest and/or which have negative sign, since these imply the greatest relocation compared to the centroid. It can often happen, for instance, that apart from one or two points, the weights are all close to +1, indicating that the significant differences are concentrated in a few points, but that the remaining structure of the configuration closely resembles the centroid. (This incidentally, could never be detected using a dimensional model where all point co-ordinates are *ex hypothesi* systematically weighted.)

In P3 the vector weights are comparable across individual configurations (this is not true of P4) and this provides a second important type of comparison. Presumably, the stimulus points where vector weights vary most from configuration to configuration are the ones which are least stable in the configuration and could be removed from analysis, or alternatively could be given more detailed study. It often happens that the variation in weights for a given stimulus vector is higher in some individual configuration than in others, which suggests that variation is concentrated in a particular area or substructure of the configuration. Once again such a difference cannot be detected using the dimensional models.

In many applications, the simple perspective model P3 shows a dramatic increase in explained variation compared to the basic model (P0), and to P1*, and provides a considerable degree of detail for analysis. The P3 model is further illustrated in Figure 7.8. Here the eight crimes configuration of Figure 7.4b is taken as the centroid configuration. Two sets of vector weights are then applied to produce the 'private spaces' of (b) and (c) in Figure 7.8. As in the INDSCAL and P1 models, the overall shape of the configuration is changed, but in P3 the local structure is also changed, as any cluster analysis would dramatically show!

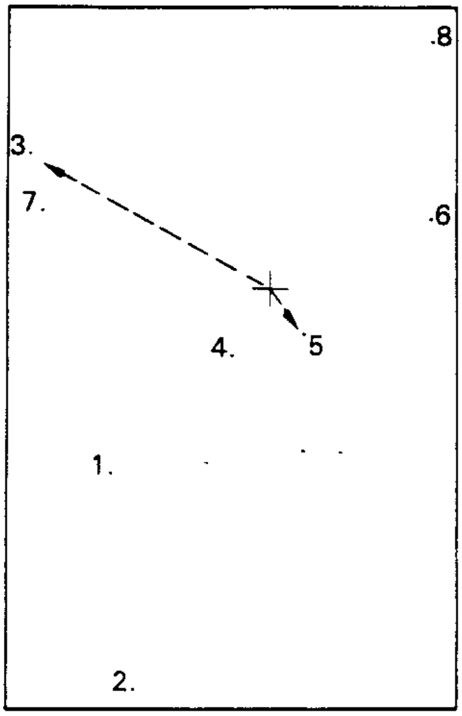
The main differences between (b) and (c) in their *pattern* of weighting are concentrated in the location of Rape (2) and Libel (5). (b) isolates Rape further from assault (moving it away in a south-westerly direction from all the other points) and (c) projects Rape into the opposite direction entirely, to join receiving as its nearest

*Such a result must be treated with some caution, for P3 allows one parameter to be estimated for each stimulus, whereas P1 simply allows one for each dimension. Usually the number of stimuli considerably exceeds the number of dimensions, hence one would expect a better fit for P3 compared to P1. The question is just how dramatic an increase is needed before deciding that it is not simply due to the additional degrees of freedom. See Langeheine (1980) for information relevant to this decision.

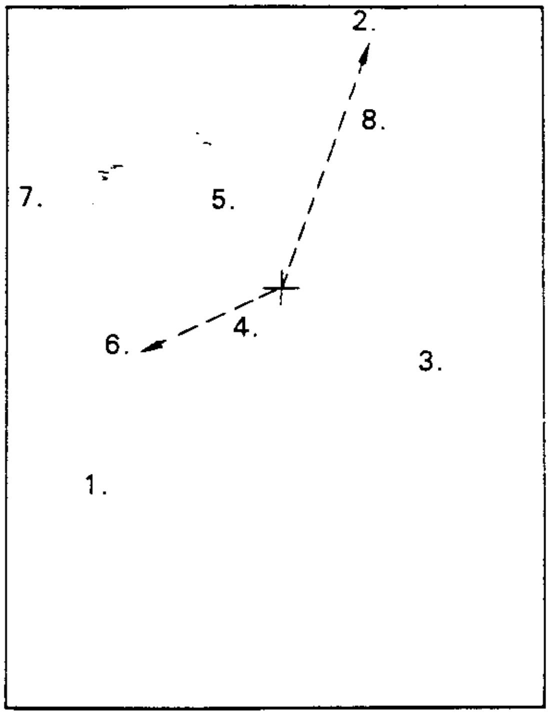


(a) Original configuration (Fig 7.4 b)

PRIVATE SPACES (Fixed origin; dotted vector denotes negative weight)



(b) V_i = (0.8,1.3,−1.1,1.5,
−0.9,0.5,0.8,0.8)



(c) V_i = (0.9,−1.2,0.7,0.9,1.5,
−0.5,0.8,0.5)

Figure 7.8 *P3: perspective model (fixed origin)*

neighbour. However, whilst the size of the weight for libel differs most. its effect is not so marked since libel is located fairly close to the origin in the first place and the composite effect is less dramatic. Nonetheless, (b) now locates libel more in the direction of burglary whereas (c) moves it somewhat closer to prostitution.

This example illustrates rather well the point that, when interpreting the P3 model, it is not simply the size and pattern of vector weights that are relevant but also the multiplicative effect upon the original length of the vector. A massive weight on a point located close to the origin can often move a point a very small distance, whereas it only needs a fairly small weight to move a peripheral point yet further away.

With some justice, the P3 model has been hailed as the major innovation introduced into MDS by PINDIS. It certainly provides a powerful and subtle form of analysis of individual differences and often gives insight into the detail about the source of variation in configurations.

7.4.3.2 P4, the idiosyncratically translated and weighted vector model

In the words of all good detective stories, 'a little thought should convince the reader' that vectors drawn from different origins will alter the pattern and shape of the configuration. The facts of the matter are illustrated in Figure 7.9. Here the centroid configuration consists of three stimulus points (labelled 1, 2 and 3) which form an equilateral triangle centred upon the origin (0, 0). Initially, the vector lengths drawn from the origin of the centroid configuration are all unity. Suppose now that three individuals all happen to employ the *same* vector weighting, namely ($\frac{1}{2}$, -1, $1\frac{1}{2}$). (In the normal way, P4 model individual vector weights can, and will, differ. Making them identical just simplifies the example.)

From the perspective *located at the origin* (labelled *A*), the original equilateral triangle will be deformed by the weights into the (1^a, 2^a, 3^a) triangle joined by the unbroken line. But from the perspective of B at (-2, 2), the configuration (1^b, 2^b, 3^b), denoted by the dotted line, looks quite different, and it looks different again from *C*'s perspective at (-1, 2). Convince yourself that the differences between *A*'s, *B*'s and *C*'s triangles arise purely from the fact that they have different origins. (The configurations would be different again if they did not share the same vector weights.)

For obvious reasons, P4 is often called the 'points of view' model (though this term was originally used by Tucker to refer to a quite different model.) In P4, it is the idiosyncratic origins that can be directly compared and that form the main focus of attention. For instance, the question of whether two subgroups of subjects differ significantly in their perspectives can be readily investigated. But the individual vector weights cannot be directly compared *except* in conjunction with the idiosyncratic shift of origin, which would mean constructing a new set of vectors all emanating from the same origin.

There have been few studies to date which have used this model, and only one where the increase in explained variations is impressive (Lingoes 1977).

7.4.4 Variants and options within PINDIS

As currently programmed, the user may also use PINDIS in an 'external' manner by inputting a hypothesis configuration instead of calculating the centroid. If so requested, this configuration will *not* be differentially rotated (for the dimensional models) and the origin will *not* be translated (for the vector models). The net effect is to suppress P2 and P4 respectively; by a separate user-controlled parameter, P5 can also be suppressed.

The external use of PINDIS is most appropriate when a target configuration is being used as a fixed reference point for other configurations (as in replications and confirmatory studies) and where a known (or hypothesised) structure such as physical properties or a geographical map underlies the data. Note that a hypothesis matrix can be input (to replace the centroid configuration) without also requiring that it be fixed.

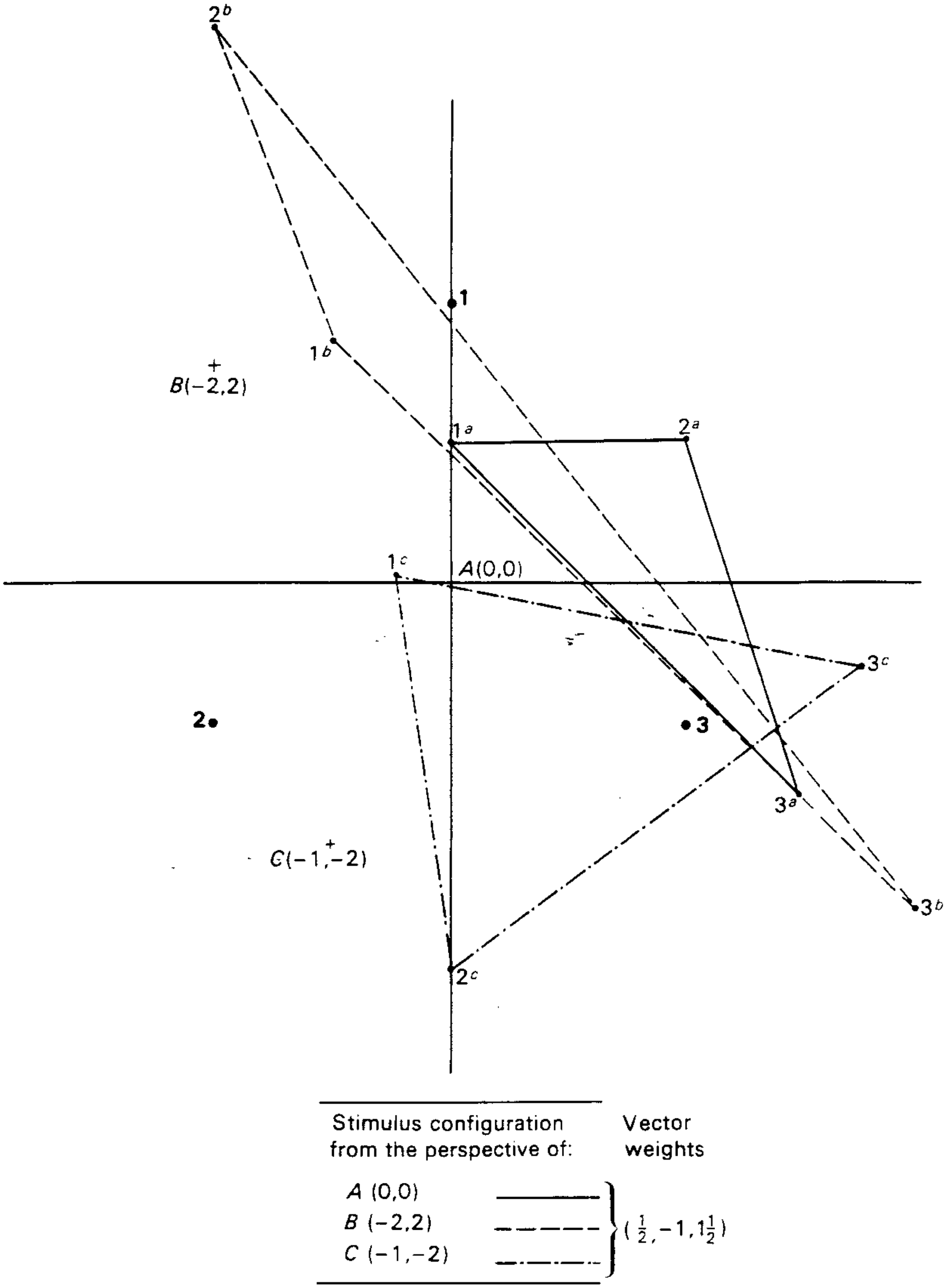


Figure 7.9 *P4: perspective model (idiosyncratic origins)*

Finally, the user may choose to have a specific stimulus location as the origin of the centroid configuration for the vector models. This is especially useful if the configurations being compared have a radex structure, with some stimulus item forming the natural origin. Details of the program parameters used to implement these options are given in MDS(X) *User's Manual* documentation for PINDIS.

7.4.5 An application of PINDIS

A range of published examples of the application of PINDIS is included in Table 7.4. It is interesting to note that the 'optimal' PINDIS model often turns out to be either the simple dimensional model (P1) or the simple vector model (P3). Only the somewhat atypical second example of Lingoes (1977) gives strong support to the double weighting model.

These examples also show that PINDIS can be used in an exploratory or a confirmatory mode, or in combination of both.

A further example, also drawn from the occupational cognition study, illustrates these points. A set of 48 subjects of varying status and occupational backgrounds were asked to rate or rank 16 occupational titles on three criteria frequently employed by sociologists to serve as forms of 'prestige', to which were added two factual criteria concerned with estimated average income and how much the subject thought she knew about the occupations concerned.

The data were scaled internally using MDPREF with the following values of an overall goodness of fit measure (linear root mean square correlations) of the data to the 3-D MDPREF solution. (Data, solutions and further details are contained in Coxon and Jones 1979, pp. 86–105):

Criterion		Goodness of fit RMS, 3-D MDPREF solution with data
1. *Social standing*	(own opinion of general standing in the community)	0.90
2. *Prestige and rewards*	(an occupation ought to get)	0.89
3. *Social usefulness*		0.84
4. *Monthly earnings*	(estimated by subject)	0.87
5. *Cognitive distances*	(how much subject knows about what a job involves)	0.73

We were interested in knowing how these MDPREF solutions differed among themselves, and also how they compared to the INDSCAL group space configuration (obtained by separate estimation of similarity and discussed earlier under 4.6 and illustrated in Figure 4.5). The differences among the MDPREF configurations were first investigated by using a straightforward PINDIS analysis. Their similarity to the 3-D INDSCAL group space was then analysed by using it as a fixed hypothesis configuration in a second run.

The result of the first analysis is summarised in Table 7.5, and the centroid configuration is given in Figure 7.10. The resemblance between this and the INDSCAL configuration of Figure 4.5 is, at least at first sight, very marked indeed. In terms of similarities between the configurations (Table 7.5a), there is quite good internal agreement: on average 69 per cent of variation between the configurations is attributable to admissible transformations, with the two status criteria (1 and 2) being especially well fit and estimated earnings (4) particularly badly fit. Neither of the dimensional models improves this fit in any way at all—a mere 2 per cent increase is involved. On the other hand the vector models do rather better, with 17 per cent improvement for P3 and 24 per cent improvement for P4. Once again, the earnings configuration fits relatively poorly, but its fit is markedly improved by allowing it to have an idiosyncratic origin.

Reference	'Subjects' Configurations	Stimuli	PINDIS models P0 P1 P2 P3 P4 P5	Fit (r^2) P0	Fit (r^2) 'Optimal'	Comments
	(i) 2 species-related, but apparently dissimilar, fish structures. *Lingoes 1977*	29 points defining fish outline	√ √ — √ — √	0.72	P5:0.96	Dimensional weighting does better than vector despite fewer parameters. One of the few instances where P5 produces markedly better fit.
	(ii) 3 biological (genetic) 'maps' measuring dermaglyphic, anthropometric and genetic characteristics of the Indian population.	7 Latin American villages	√ √ √ √ — √	0.63	P3:0.90	Interesting use of target (geographical) configuration which is *not* kept fixed. Clear superiority of P3 over P1 suggests interesting conclusions (q.v.).
Borg 1980 (i) *data* Lingoes et al. (1979)	14 individual subjects (SSAR-1)	6 Political Party × 6 'Closeness' combinations	√ √ √ √ — √	0.77	P3:0.88	Not a marked improvement.
(ii) *data* Green and Rao (1972)	41 individual subjects (MDSCAL)	15 breakfast foods	√ √ √ √ — —	0.72	P3:0.85	Lot of individual variability.
(iii) *data* Lingoes et al. (1979)	10 normal (N) and 4 colour-dificient (CD) subjects (Classic Scaling)	10 colour tiles	√ √ √ √ — —	N:0.98 CD:0.74	(P0) P1:0.88	Z of normals used as fixed target for assessing colour-deficients. As expected, simple dimensional weighting of colour-circle is best model.
Maimon et al. 1980	4 groups of managerial graduates and supervisors. (SSA-1, 2D)	9 items (facet-design) type × area of skills	√ √ √ √ — √	0.71	P3:0.90	High variability on all other models.
Coxon and Jones 1980	6 individual deviant cases from occupational cognition project (MINISSA, 3D)	16 occupations	√ √ — √ — —	0.29	P3:0.70	Used fixed target (INDSCAL) configuration.

Table 7.4 *Applications of* PINDIS

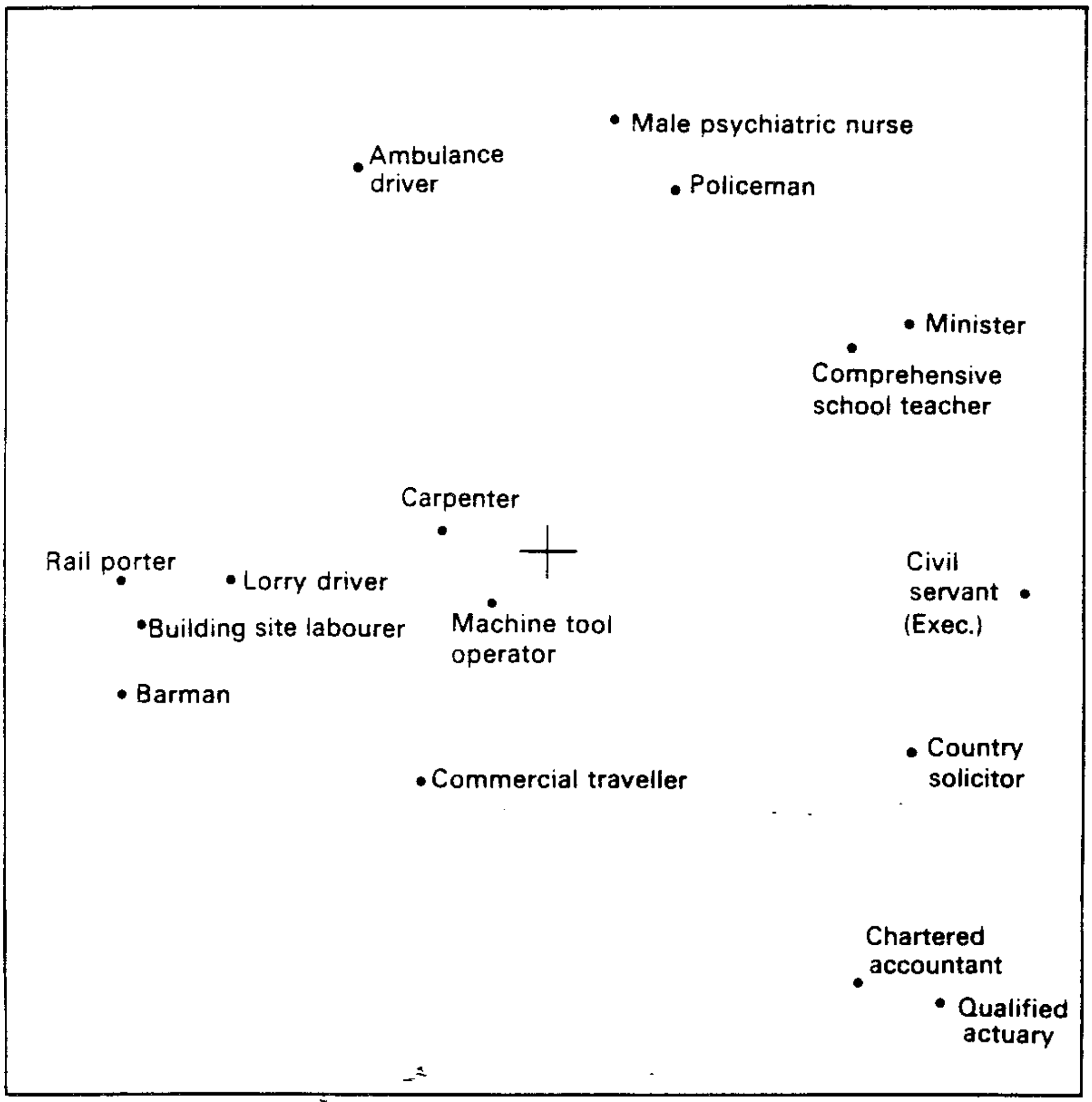


Figure 7.10 *First 2 dimensions of PINDIS centroid configuration derived from 5 MDPREF configurations*

Concentrating now on P3, where are the discrepancies located? An analysis of the vector weights indicates unequivocally that the first three status criteria and the last two ('cognitive') criteria agree very considerably among themselves, and contrast with each other in terms of the relative size and pattern of the weights. Most 'unscrambling' is done with respect to the machine tool operator (the prestige configuration has a vector weight of 2.43, whilst the cognitive distance configuration has one of the few negative weights of −0.17 for this occupation), and to a lesser extent the civil servant (the earnings configuration attaches a weight four or five times that of the other criteria).

In the second run, the configurations were related to the INDSCAL 3-D configuration—which, as we have commented, the PINDIS centroid configuration seems to resemble closely—the degree of fit drops dramatically (see Table 7.5b). Clearly there *are* very considerable differences with respect to the INDSCAL configuration and we have been over-impressed by their surface resemblance. Nor is the fit improved by the dimensional model. Since the INDSCAL configuration is fixed,

Configuration	P0	P1	P2	P3	P4	P5
		Communalities (r^2) for PINDIS Transformations				
	Basic	Dimensional weighting	Dim. weighting and rotation	Vector weighting	Vector weighting and translations	Double weighting
1 Social Standing	0.83	0.83	0.83	0.94	0.99	0.88
2 Prestige and Rewards	0.83	0.84	0.84	0.96	0.92	0.90
3 Social Usefulness	0.77	0.78	0.78	0.86	0.92	0.83
4 Monthly Earnings	0.40	0.42	0.44	0.68	0.89	0.81
5 Cognitive Distance	0.62	0.64	0.64	0.82	0.93	0.73
Average	0.69	0.70	0.71	0.86	0.93	0.83

(a) Full analysis (all simple models, no hypothesised configurations)

	P0	P1	P2	P3	P4	P5
1 Social Standing	0.26	0.27	—	1.00	—	0.64
2 Prestige and Rewards	0.16	0.17	—	0.79	—	0.65
3 Social Usefulness	0.19	0.20	—	0.55	—	0.47
4 Monthly Earnings	0.12	0.12	—	0.40	—	0.57
5 Cognitive Distance	0.25	0.26	—	0.62	—	0.59
Average	0.19	0.20	—	0.67	—	0.58

(b) Analysis with INDSCAL 3-D configuration as hypothesis

Table 7.5 PINDIS *analysis of 5 occupational criteria configurations* the analysis excluded rotation of axes and translation of origin, so the parameters of P2 and P4 were not estimated. Once again, the simple vector model does a dramatic job of improvement, with the social standing configuration now fitting perfectly, and the earnings configuration still being rather poorly fit.

Such a finding is substantively important: it shows for instance that we can conclude that data obtained from judgments of 'general standing' of an occupation give rise to a conclusion that is a systematic transformation of the INDSCAL structure obtained from judgment about 'overall similarity'. At the very least, such a finding strongly contests the received opinion that the latter is a cognitive criterion and is generically distinct from the evaluative status criteria.*

7.5 Preference Mapping (PREFMAP I and II)

The Carroll-Chang (Carroll 1972, p. 114 et seq.) preference mapping models form a hierarchy of models, akin in many ways to PINDIS. They differ principally in that PREFMAP (PM) is designed primarily for *external* scaling (where the user provides a stimulus configuration) and the input data consist of a rectangular matrix of ratings or rankings of p stimuli given by N subjects. The purpose is to map *each subject* into the stimulus space in the form of an ideal point (PM Phases I–III) or as a vector (PM Phase IV) according to a hierarchy of increasingly complex models. (The transformation may be linear/metric or quasi non-metric/ordinal according to the user's choice.) We have already encountered earlier in this book the two simplest models: the *simple distance model* (Phase III) (5.3.3.1 and 6.2.4) and the *vector model* (Phase IV) (5.3.2 and 6.2.2) and so the focus here will be on the more complex models—the rotated and weighted distance model (Phase I, akin to P2 in PINDIS) and the weighted distance model (Phase II, akin to P1 and INDSCAL). But it will be helpful to begin by summarising the full range of the PREFMAP models.

7.5.1 The PREFMAP hierarchy of models

The basic notion of all the models in PREFMAP is that an individual's preference ranking is a function of the distance separating her point of maximum preference (ideal point) and the stimulus points. It is the way in which the distances are defined which differentiates the models.† (PREFMAP can also be seen as extending and generalising Coombs' unfolding model, and as showing that the vector model of preference is a special case of unfolding.)

Briefly, the hierarchy of models is as follows:

I *General Unfolding Model (PM1)*

This is the most general model. Each subject is viewed as having a specific, most-

*When a two-dimensional PINDIS analysis is performed, the results are much the same, but even more marked. Admissible transformations account for 75% of variation in the 'internal' PINDIS analysis and for 14% variation when related to the 'external' INDSCAL configuration. In neither case do the dimensional models add more than a derisory amount (2% in both cases), but the increase due to simple vector weighting is more impressive (18%, and 57% in the external case). But the 'unscrambling' is different in detail: the machine tool operator is still relocated more than any other occupation, but the actuary and accountant are also seen to be relatively unstable in their positioning.

†Technical details of the models are given in Carroll (1972) and in Coxon and Jones (1979, p. 106 et seq.), and a detailed exposition of the computational procedure and output details is given in van Schuur (1977).